FINITE DEFICIENCY INDICES AND UNIFORM REMAINDER IN WEYL'S LAW

LUC HILLAIRET

Introduction

Von Neumann theory classifies all self-adjoint extensions of a given symmetric operator A in terms of the so-called *deficiency indices*. Since the choice of the self-adjoint condition reflects the physics that is underlying the problem, it is natural to ask how the spectra of two different self-adjoint extensions A_0 and A_1 can differ.

In some cases, the deficiency indices are finite. This happens for instance in the following interesting settings from mathematical physics

- Quantum graphs (see [3, 7] and section 4.1 below),
- Pseudo-Laplacians, Šeba billiards (see [5, 8], and section 4.2 below),
- Manifolds with conical singularities (see [10, 9]),
- Hybrid manifolds (see [12]).

In any of these settings we prove the following theorem (see section 2 and the applications for a more precise version)

Theorem 1. *Let A be the symmetric operator associated with one of the preceding settings. There exists a constant C such that for any self-adjoint extensions A_0 and A_1 of A we have*

$$\forall E,\ |N_1(E) - N_0(E)| \leq C$$

where N_i denotes the spectral counting function of A_i.

This fact actually derives from [4] ch. 9 sec. 3 [1]. Our proof is slightly different and based on the min-max principle but the underlying ideas are similar.

Motivation for this result came principally from [8] and the Šeba billiard setting. In this case we can take A_0 to be the standard Dirichlet Laplace operator in R, and this theorem proves that the remainder in Weyl's law is, up to a $O(1)$ term, uniform with respect to the location of the Delta potential. In the case of one delta potential the uniform bound can also be derived from the fact that the spectra of the pseudo-laplacian and the usual laplacian are interlaced (see [5] for instance).

In constrast with [3, 7, 9, 12] we consider a rather crude spectral invariant. Moreover, our result relies on the min-max principle only which is less sophisticated than the analysis performed in the former references. It should be noted, however, that our result is not a straightforward byproduct of these results and should more likely be considered as a first step. From our perspective, it

[1]We are grateful to Alexander Pushnitski for indicating to us this reference.

is quite interesting to have a general method allowing to get quite good hold on the spectral counting function before moving on to more complete spectral invariants such as heat kernel or resolvent estimates.

Acknowledgments : We are grateful to Alexander Pushnitski, Alexey Kokotov and Jens Marklof for useful comments on the first version of this note that resulted, in particular, in great improvment of the bibliography.

1. Setting and Notations

We begin by recalling some basic facts from spectral theory of self-adjoint operators as well as Von Neumann theory of self-adjoint extensions of a symmetric operator. We will use [13, 14] and [6] as references.

1.1. Basic Spectral Theory.

We consider a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$.

On $\mathcal{H}$ we consider a symmetric operator A with domain $\mathrm{dom}(A)$ and its adjoint A^*. The graph norm is defined on $\mathrm{dom}(A)$ by $\|u\|_A^2 \,=\, \|u\|^2 + \|Au\|^2$.

An operator is self-adjoint if $A = A^*$. It has compact resolvent if the injection from $\mathrm{dom}(A)$ into $\mathcal{H}$ is compact.

The spectrum of a self-adjoint operator with compact resolvent consists in eigenvalues of finite multiplicities, that form a discrete set in $\mathbb{R}$. There exists an orthonormal basis consisting of eigenvectors.

If there exists $C \in \mathbb{R}$ such that $\forall u \in \mathrm{dom}(A),\ \|Au\| \geq C\|u\|$, the operator is called *semibounded*.

For a semibounded self-adjoint operator with compact resolvent, the spectrum can be ordered into a non-decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$.

The spectral counting function is then defined by

$$N(E) \,=\, \mathrm{Card}\{\lambda_n \leq E\},$$

and by Courant-Hilbert min-max principle (see [6]) we have

$$\lambda_n \,=\, \min\left\{ \max\left\{ \frac{\langle Au, u\rangle}{\|u\|^2},\ u \in F\backslash\{0\} \right\},\ F \subset \mathcal{H}, F \text{ vector space s.t. } \dim F = n \right\}. \tag{1.1}$$

1.2. Von Neumann Theory.

This section summarizes section X.1 of [14]. We define

$$\mathcal{K}^{\pm} \,=\, \ker(A^* \mp \mathrm{id})$$
$$d_{\pm} \,=\, \dim(\mathcal{K}^{\pm}).$$

We recall that we have the following decomposition of $\mathrm{dom}(A^*)$ (see Lemma in section X.1 of [14]).

$$\mathrm{dom}(A^*) \,=\, \mathrm{dom}(\bar{A}) \oplus \mathcal{K}^+ \oplus \mathcal{K}^-,$$

and that the decomposition is orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle_{A^*}$ defined on $\mathrm{dom}(A^*)$ by

$$\forall u,v\in\mathrm{dom}(A^*),\ \langle u,v\rangle_{A^*} := \langle u,v\rangle+\langle A^*u,A^*v\rangle.$$

We denote by $\pi_\pm$ the orthogonal projection from $\mathrm{dom}(A^*)$ onto $\mathcal{K}^\pm$ and by π_0 the orthogonal projection onto $\mathrm{dom}(\bar{A})$ (so that $\pi_0 = id-\pi_+-\pi_-$).

The following theorem (theorem X.2 and corollary of [14]) provides a parameterization of all self-adjoint extensions of A

Theorem 2. *A admits self-adjoint extensions if and only if $d_+ = d_-$.*
For any self-adjoint extension A_{sa} of A, there exists a unique isometry U from $\mathcal{K}^+$ onto $\mathcal{K}^-$ such that $\forall u\in dom(A_{sa})$, $U\pi_+(u) = \pi_-(u)$.

2. The theorem

Using the notations of the preceding section we have

Theorem 3. *Let A be a symmetric operator with equal finite deficiency indices :*

$$d_+ = d_- = d<\infty.$$

Suppose that there exists A_0 a self-adjoint extension of A such that

(i) *A_0 has compact resolvent,*
(ii) *A_0 is semibounded*

Then

(1) *Any other self-adjoint extension also has compact resolvent and is semibounded.*
(2) *There exist E_0 such that, for any other self-adjoint extension A_1 the following holds*

$$\forall E\in\mathbb{R},\ |N_1(E)-N_0(E)|\leq d$$

where $N_i(E)$ denotes the spectral counting function of A_i.

As indicated in the introduction this result is actually already proved in [4] using that the difference of the resolvents $(A_0-z)^{-1}-(A_1-z)^{-1}$ is finite rank for some z.

3. Proofs

3.1. A_1 has compact resolvent.

We prove this fact by proving the stronger lemma.

Lemma 3.1. *If A_0 has compact resolvent then the injection from $dom(A^*)$ (equipped with $\|\cdot\|_{A^*}$) into $\mathcal{H}$ is compact.*

Proof. Since A_0 has compact resolvent, the injection from $(\mathrm{dom}(A_0),\|\cdot\|_{A_0})$ into $\mathcal{H}$ is compact. But $\mathrm{dom}(\bar{A})$ is closed in $\mathrm{dom}(A^*)$ for $\|\cdot\|_{A^*}$. Since $\|\cdot\|_{A_0}$ coincide with $\|\cdot\|_{A^*}$ on A_0, the injection from $(\mathrm{dom}(\bar{A}),\|\cdot\|_{A^*})$ is also compact.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{dom}(A^*)$ that is $\|\cdot\|_{A^*}$-bounded. We can extract a subsequence from $\pi_0 u_n$ that converges in $\mathcal{H}$ since the injection from $(\mathrm{dom}(\bar{A}),\|\cdot\|_{A^*})$ into $\mathcal{H}$ is compact. On $\mathcal{K}^\pm$, $\|\cdot\|_{A^*}$ is equivalent to $\|\cdot\|$ and, since $d^\pm$ are finite we can also extract convergent subsequences. This proves the Lemma. □

The Lemma says that A^* has compact resolvent and so does A_1 since A^* extends A_1 and the latter is closed.

3.2. A_1 **is semibounded.** (See also [1] sec. 85)

Let $(\lambda_k(A_0))_{k\in\mathbb{N}}$ denote the (ordered spectrum) of A_0 and consider n such that $\lambda_n(A_0)\geq 0$.

Consider $F\subset \mathrm{dom}(A_1)$of dimension $n+d$. Denote by $F_1 = F\cap \mathrm{dom}(\bar{A}) = F\cap \ker(\mathrm{id}-\pi_0)$. Theorem 2 implies that $\ker(\mathrm{id}-\pi_0)_{|\mathrm{dom}(A_1)}$ is of dimension d, and thus, since F is of dimension $n+d$, F_1 is of dimension at least n.

Moreover $F_1\subset \mathrm{dom}(\bar{A})\subset \mathrm{dom}(A_{1,0})$, thus, for all $u\in F_1$ we have

$$\langle A_1 u, u\rangle \;=\; \langle A_0 u, u\rangle.$$

Since $F_1\subset F$ it follows that

$$\max_{u\in F, u\neq 0}\frac{\langle A_1 u, u\rangle}{\|u\|^2}\geq \max_{u\in F_1, u\neq 0}\frac{\langle A_0 u, u\rangle}{\|u\|^2}.$$

Since $\dim F_1\geq n$ and $\lambda_n\geq 0$, it follows from the min-max principle that the right-hand side is non-negative.

We thus obtain, that for all $F\subset \mathrm{dom}(A_1)$ of dimension $n+d$ we have

$$\max_{u\in F, u\neq 0}\frac{\langle A_1 u, u\rangle}{\|u\|^2}\geq 0.$$

This implies that A_1 has at most $n+d-1$ negative eigenvalues. (otherwise the subspace generated by $n+d$ negative eigenvalues would contradict the preceding bound).

3.3. **Comparing** N_0 **and** N_1**.** According to the previous section we have that A_1 also is semibounded with compact resolvent so that we can denote by $(\lambda_k(A_1))_{k\in\mathbb{N}}$ its ordered spectrum. We also denote by V_n^i the vector space generated by the first n eigenvectors of A_i.

For any n, set $F = V^1_{n+d}\cap \mathrm{dom}(\bar{A})$. Making the same reasoning as previously we find that F is of dimension at least n and

$$\max_{u\in V_{n+d}, u\neq 0}\left\{\frac{\langle A_1 u, u\rangle}{\|u\|^2}\right\}\;\geq\;\max_{u\in F, u\neq 0}\left\{\frac{\langle A_0 u, u\rangle}{\|u\|^2}\right\}.$$

By definition of V_{n+d} the left-hand-side is $\lambda_{n+d}(A_1)$ and, using the min-max principle, the right-hand side is bounded below by $\lambda_n(A_0)$. Thus, we obtain,

$$\forall n,\;\lambda_{n+d}(A_1)\geq \lambda_n(A_0).$$

Observe that $N_0(E)$ is characterized by

$$\lambda_{N_0(E)}(A_0)\leq E<\lambda_{N_0(E)+1}(A_0)$$

Using this and the preceding inequality we find that, for all E we have

$$E\leq \lambda_{N_0(E)+d+1}(A_1)$$

An thus, for all E we have

$$N_1(E)\leq N_0(E)+d.$$

Since A_0 and A_1 now play symmetric roles we also have

$$\forall n,\ \lambda_{n+d}(A_0) \geq \lambda_n(A_1).$$

and thus

$$N_0(E) \leq N_1(E) + d.$$

The final claim of the theorem follows.

4. Applications

4.1. **Quantum graphs.** Quantum graphs are now well-studied objects from mathematical physics (see [11] for an introduction). A very rough way of defining a (finite) quantum graph is the following.

Pick K positive real numbers (the lengths), set $\mathcal{H} = \bigoplus_{i=1}^K L^2(0, L_i)$ and $\mathcal{D} = \bigoplus_{i=1}^K \mathcal{C}_0^\infty(0, L_i)$ and define A on $\mathcal{D}$ by $A(u_1 \oplus u_2 \oplus \cdots \oplus u_K) = -(u_1'' \oplus u_2'' \oplus \cdots \oplus u_K'')$. This operator is symmetric. Any self-adjoint extension of A is called a quantum graph.

Remark 4.1. Usually quantum graphs are constructed starting from a combinatorial graph given by its edges and vertices. This combinatoric data is actually hidden in the choice of the self-adjoint condition.

One basic question is to understand to which extent the knowledge of the spectrum determines the quantum graph (i.e. the lengths and the boundary condition).

It is known that there are isospectral quantum graphs [2] and the following theorem says that, as far as counting function is concerned it is quite difficult to determine the self-adjoint condition.

Theorem 4. *For any quantum graph with K edges the following bound holds :*

$$\left| N(E) - \frac{\mathcal{L}}{\pi} E_+^{\frac{1}{2}} \right| \leq 3K$$

where $\mathcal{L} := \sum_{i=1}^K L_i$ and $E_+ := \max(E, 0)$.

Proof. Fix K and the choice of the lengths. We choose one particular self-adjoint extension A_D that consists in decoupling all the edges and putting Dirichlet boundary condition on each end of each edge. The spectrum is then easily computed and we have

$$\operatorname{spec}(A_D) = \bigcup_{i=1}^K \left\{ \frac{k^2\pi^2}{L_i^2},\ k \in \mathbb{N} \right\}.$$

In particular we have (denoting by N_D the counting function of the Dirichlet extension)

$$N_D(E) = \sum_{i=1}^K \left[\frac{L_i}{\pi} E_+^{\frac{1}{2}} \right],$$

where $[\cdot]$ denotes the integer part. In particular, we have

$$\left| N_D(E) - \frac{\mathcal{L}}{\pi} E_+^{\frac{1}{2}} \right| \leq K.$$

We now compute the deficiency indices. A straightforward computation yields $d_+ = d_- = 2K$. And thus, using the main theorem and triangular inegality we obtain that for any quantum graph

$$\left| N_D(E) - \frac{\mathcal{L}}{\pi} E_+^{\frac{1}{2}} \right| \leq 3K,$$

indepently of the choice of lengths. □

4.2. **Pseudo-Laplacian with Delta potentials.** It is known that on Riemannian manifolds of dimension 2 or 3 it is possible to add so-called Delta potentials. From a spectral point of view this corresponds to choosing a finite set of points P and to consider the Riemannian Laplace operator defined on smooth functions with support in $M\backslash P$. (see [5] for instance)

Remark 4.2. This construction is also possible starting from a bounded domain in $\mathbb{R}^2$ with, say, Dirichlet boundary condition. We obtain the so-called Šeba billiards (See [8] for instance)

There are several self-adjoint extensions and (a slight generalization of) Lemma of [5] proves that, in this setting the deficiency indices are $d :=$ card P. Following Colin de Verdière we call any such self-adjoint extension a *Pseudo-Laplacian with d Delta potentials*

Application of the theorem gives the following.

Theorem 5. *Let M be a closed Riemannian manifold of dimension 2 or 3. Let N_0 be the counting function of the (standard) Laplace operator on M. For any pseudo-laplacian with d Delta potential the following holds*

$$|N(E) - N_0(E)| \leq d$$

It should be noted first that the bound depends only on the number of Delta potentials and not on their location, and second that the effect of adding Delta potentials is much smaller than the usual known remainder terms in Weyl's law for $N_0(E)$.

4.3. **Others.** There are two other settings where finite deficiency indices occur that are worth mentioning. In both case one could apply the theorem to get a Weyl's asymptotic formula independent of the choice of the self-adjoint condition up to a $O(1)$ term. These are

(1) Manifolds with conical singularities. The common self-adjoint extension in use corresponds to Friedrichs extension and if one changes the self-adjoint extensions at the conical points (see [10, 9]) then the counting function is affected only by some bounded correction. Observe that the deficiency index associated with the Laplace operator on the cone of opening angle α is $2[\frac{\alpha}{2\pi}] - 1$ (with $[\cdot]$ the integer part).
(2) The so-called *hybrid manifolds* that are studied in [12] which are obtained by, in some sense, grafting quantum graphs onto higher dimensional manifolds. Here again, one can compare the counting function of the chosen self-adjoint extension to the natural one which is obtained when all the parts are decoupled.

References

[1] N.I Akhiezer and I.M Glazman. *Theory of linear operators in Hilbert space. Vol. II.* Monographs and Studies in Mathematics, 10. Pitman, Boston, 1981.

[2] R. Band, O. Parzanchevski and G. Ben-Shach. The isospectral fruits of representation theory: quantum graphs and drums. *J. Phys. A*, 42(17):175–202, 2009.
[3] J, Bolte and S. Endres. The trace formula for quantum graphs with general self adjoint boundary conditions. *Ann. Henri Poincaré*, 10(1):189–223,2009.
[4] M.S. Birman, M.Z. Solomjak. *Spectral Theory of Self-Adjoint Operators in Hilbert Space* Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, 1987.
[5] Y. Colin de Verdière, Yves. Pseudo-laplaciens I. *Ann. Inst. Fourier* 32(3):275–286,1982.
[6] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I.* Interscience Publishers, New York, 1953.
[7] J.M. Harrison and K. Kirsten. Vacuum energy, spectral determinant and heat kernel asymptotics of graph Laplacians with general vertex matching conditions. preprint, arXiv:0912.0036v1, 2009.
[8] J. P. Keating, J. Marklof and B. Winn. Localised eigenfunctions in Šeba billiards. preprint, arXiv:0909.3797v1, 2009.
[9] K. Kirsten, P. Loya, Paul and J. Park, Jinsung. Exotic expansions and pathological properties of ζ-functions on conic manifolds. *J. Geom. Anal.* 18(3):835–888, 2008.
[10] A. Kokotov. Compact polyhedral surfaces of an arbitrary genus and determinants of Laplacians. *preprint*, arXiv:0906.0717v1, 2009.
[11] P. Kuchment. Quantum graphs: an introduction and a brief survey. Analysis on graphs and its applications, 291–312, Proc. Sympos. Pure Math., 77, Amer. Math. Soc.,Providence , 2008.
[12] K. Pankrashkin, S. Roganova and N. Yeganefar. Resolvent expansions on hybrid manifolds *preprint*, arXiv:0911.3282v1, 2009.
[13] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional Analysis.* Academic Press, Inc., New York, 1980. Second Edition.
[14] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness..* Academic Press, Inc., New York, 1975.

E-mail address: Luc.Hillairet@math.univ-nantes.fr

UMR CNRS 6629-Université de Nantes, 2 rue de la Houssinière,, BP 92 208, F-44 322 Nantes Cedex 3, France